Nonlocality without inequality for spin-s system

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Abstract

We analyze Hardy's non-locality argument for two spin-s systems and show that earlier solution in this regard was restricted due to imposition of some conditions which have no role in the argument of non-locality. We provide a compact form of non-locality condition for two spin-s systems and extend it to n number of spin-s particles. We also apply more general kind of non-locality argument still without inequality, to higher spin system.

Introduction

The well known contradiction between quantum theory and local realism was first revealed by Bell's inequality [1]. Interestingly Hardy gave a proof of non-locality without using inequality for two spin-$\frac{1}{2}$ particles [2] and all pure entangled states excepting maximally entangled one, exhibit this kind of non-locality. This result has been generalized for many qubits and higher dimensional bi-partite system [3, 4, 5]. Kar showed there is no two qubits mixed state which satisfy Hardy's non-locality argument for two observabls on each setting [6] and Cabello gave another argument of Bell's theorem without inequalities for GHZ and W state [7]. Based on Cabello's logic structure Liang et. al [8] provided an example for two qubit mixed state which shows non-locality still without inequality.

Hardy's argument has been generalized for two spin-s particles by Clifton and Niemann [9]. Based on Clifton and Niemann's argument, Ghosh et.al [10] showed that for any two spin-s particles and two measurement possibilities for each particle, there are infinitely many states (called "Hardy states") showing Hardy's non-locality. They also showed that

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for two spin-s particles and for given choice of observables, the closure of the set of states showing Hardy-type non-locality constitute a 2s dimensional subspace of the Hilbert space associated with the system.

In this paper we show that for the case of two spin s particles the set of solutions found in Ghosh.et.al.[10], is restricted due to use of some constraints which have no role in the non-locality argument. In this paper we shall show that abandoning the redundancy will provide a larger set of solutions whose closure will constitute a $4s^2$ dimensional subspace instead of $2s$ dimensional subspace [10] of the Hilbert space associated with the system. We also provide a new argument of non-locality without inequality for two spin-s particle along the procedure (based on Cabello's logic structure) followed by Liang et.al.[8] where Hardy's argument is the extreme case of this more generalized argument.

Hardy's non-locality for two spin-s system

Let us consider two spin-s ($s = \frac{1}{2}, 1, \frac{3}{2}.......$) particles A and B. Let s_a and $s_{a'}$ represents spin component of particle A along a and a' direction. Similarly s_b and $s_{b'}$ represents spin component of particle B along b and b' direction. The value of s_a, s_b, $s_{a'}$, $s_{b'}$ runs from $-s$ to $+s$.

Following Clifton and Niemann's procedure a state $|\psi\rangle$ is called a Hardy state if the following condition are satisfied.

$$P(s_a = s_b = s) = 0 \tag{1}$$

$$P(s_a + s_{b'} \geq 0) = 1 \tag{2}$$

$$P(s_{a'} + s_b \geq 0) = 1 \tag{3}$$

$$P(s_{a'} = s_{b'} = -s) = p \tag{4}$$

One can easily check that equation (1) - (4) are incompatible with local realism. Interestingly if we express the above equations in terms of probability on one dimensional projector, one can easily see that some equations have no role in constructing the argument to show the contradiction with local realism. To make it explicit we first consider the system of two particles, both having spin-1. For $s = 1$ the above equations can be written as

$$P(s_a = +1, s_b = +1) = 0 \tag{5}$$

$$P(s_a = -1, s_{b'} = -1) = 0 \tag{6}$$

$$P(s_a = -1, s_{b'} = 0) = 0 \tag{7}$$

$$P(s_a = 0, s_{b'} = -1) = 0 \tag{8}$$

$$P(s_{a'} = -1, s_b = -1) = 0 \tag{9}$$

$$P(s_{a'} = -1, s_b = 0) = 0 \tag{10}$$

$$P(s_{a'} = 0, s_b = -1) = 0 \tag{11}$$

$$P(s_{a'} = -1, s_{b'} = -1) = p \tag{12}$$

To show that the above equations contradict local realism we start from equation (12). This equation tells that if there is Hidden Variable Theory (HVT) then there is some hidden variable states for which $s_{a'} = -1$, $s_{b'} = -1$. We consider one such state denoted by λ (say). Now for this state λ equations (6) and (8) tell $s_a = +1$. Similarly for those λ equations (9) and (10) tell $s_b = +1$. Then $P(s_a = +1, s_b = +1)$ should have been non-zero for all λ states which satisfy (12) but this contradicts equation (5).

One should note that to run the Hardy's argument for $s = 1$, we have not used equations (7) and (11). Hence Hardy's state need not be orthogonal to the projectors $P[|s_a = -1, s_{b'} = 0\rangle]$ and $P[|s_{a'} = 0, s_b = -1) = 0\rangle]$ appearing in (7) and (11). It has to be orthogonal only to the projectors $P[|s_a = +1, s_b = +1\rangle]$, $P[|s_a = -1, s_{b'} = -1, \rangle]$, $P[|s_a = 0, s_{b'} = -1\rangle]$, $P[|s_{a'} = -1, s_b = 0\rangle]$, and $P[|s_{a'} = 0, s_b = -1\rangle]$.

Discarding all the unnecessary restrictions the generalized Hardy's argument for two spin-s system takes the following form.

$$P(s_a = s_i, s_b = s_j) = 0 \tag{13}$$

$$P(s_a \neq s_i, s_{b'} = s_l) = 0 \tag{14}$$

$$P(s_{a'} = s_k, s_b \neq s_j) = 0 \tag{15}$$

$$P(s_{a'} = s_k, s_{b'} = s_l) = p \tag{16}$$

Where s_i, s_j, s_k and s_l can take values from $-s$ to $+s$. For a fixed value of s_i, s_j, s_k and s_l each of the sets (14) and (15) contains $2s$ equations. So the total number of equations in (13), (14) and (15) is $(4s + 1)$.

To satisfy these $(4s + 1)$ equations , Hardy's state has to be orthogonal to the $(4s + 1)$ linearly independent vectors corresponding to the projections appearing in equations (13), (14) and (15). Let M be the closed subspace generated by these $(4s + 1)$ vectors and let $\overline{M}$ be the orthogonal complement to M. The dimension of $\overline{M}$ is obviously $4s^2$. Let M' be the closed subspace generated by the above $(4s + 1)$ vectors together with the vector corresponding to the projection operators appearing in equation (16), and let $\overline{M'}$ be the orthogonal complement to M'. Obviously $\overline{M'}$ is of dimension $(4s^2 - 1)$. Then any member of the subset $\overline{M} - \overline{M'}$ (of the Hilbert space associated with the system), whose closure is of the dimension $4s^2$, is a solution of equation $(13) - (16)$.

Generalization to n spin-s particles

The above non-locality argument can easily be extended to n $(n \geq 2)$ number of spin-s system. Let s_{a_k} and $s_{a'_k}$ represent spin component of k th particle along the vector a and

a' respectively. The value of s_{a_k} and $s_{a'_k}$ runs from $-s$ to $+s$. Then Hardy's non-locality conditions for n $(n \geq 2)$ number of spin-s system is given by

$$P(s_{a_1} = s_i^1, s_{a_2} = s_i^2, \ldots \ldots s_{a_n} = s_i^n) = 0 \tag{17}$$

$$P(s_{a_1} \neq s_i^1, s_{a'_2} = s_j^2, \ldots \ldots s_{a'_n} = s_j^n) = 0 \tag{18}$$

$$P(s_{a'_1} = s_j^1, s_{a_2} \neq s_i^2, s_{a'_3} = s_j^3 \ldots \ldots s_{a'_n} = s_j^n) = 0 \tag{19}$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$P(s_{a'_1} = s_j^1, s_{a'_2} = s_j^2, \ldots \ldots s_{a'_{(n-1)}} = s_j^{(n-1)}, s_{a_n} \neq s_i^n) = 0 \tag{20}$$

$$P(s_{a'_1} = s_j^1, s_{a'_2} = s_j^2, \ldots \ldots s_{a'_n} = s_j^n) = p \tag{21}$$

Where s_i^n and s_j^n can take any value from $-s$ to $+s$. We have already checked the case where $n = 2$. Similar kind of argument will show that equations $(17) - (21)$ contradict local realism. The states which satisfy above equations has to be orthogonal to $(2ns + 1)$ linearly independent vectors corresponding to the projection operators appearing in equations $(17) - (20)$. Let M be the closed subspace generated by these $(2ns + 1)$ vectors and let $\overline{M}$ be the orthogonal complement to M. The dimension of $\overline{M}$ is obviously $[(2s + 1)^n - (2ns + 1)]$. Let M' be the closed subspace generated by the above $(2ns + 1)$ vectors together with the vector corresponding to the projection operators appearing in equation (21), and let $\overline{M'}$ be the orthogonal complement to M'. Obviously $\overline{M'}$ is of dimension $[(2s + 1)^n - (2ns + 2)]$. Then any member of the subset $\overline{M} - \overline{M'}$ (of the Hilbert space associated with the system), whose closure is of the dimension $[(2s+1)^n - (2ns+1)]$, is a solution of equation $(17) - (21)$.

Cabello-type nonlocality argument for two spin-s system

The logic structure of Hardy's nonlocality goes as follows: A and B sometimes happen, A always implies D, B always implies C, but C and D never happen. Cabello gave another logic structure to prove the Bell's theorem without inequality for GHZ and W state [7]. Cabello's logic structure is like that: A and B sometimes happen, A always implies D, B always implies C, but C and D happen with lower probability than A and B. Here we will give a cabello like non-locality argument for two spin-s system. Let us first consider the case for $s = 1$. We rewrite the equations $(13) - (16)$ with q (where $0 < q < p$) on the right hand side of equation (13) and where for $s_i = s_j = +1$ and $s_k = s_l = -1$

$$P(s_a = +1, s_b = +1) = q \tag{22}$$

$$P(s_a = -1, s_{b'} = -1) = 0 \tag{23}$$

$$P(s_a = 0, s_{b'} = -1) = 0 \tag{24}$$

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$$P(s_{a'} = -1, s_b = -1) = 0 \tag{25}$$

$$P(s_{a'} = -1, s_b = 0) = 0 \tag{26}$$

$$P(s_{a'} = -1, s_{b'} = -1) = p \tag{27}$$

One can check that the above equations contradict local realism when $q < p$. To show that contradiction, we consider the hidden variable states λ for which $s_{a'} = -1$, $s_{b'} = -1$. Now for those states λ equations (25) and (26) tell that the value of s_b must be $+1$. Again for these λ equations (23) and (24) tell that the value of s_a must be $+1$. So $P(s_a = +1, s_b = +1)$ should be at least p, which contradict equation (22) as $q < p$. Instead of putting q (where $q < p$) in equation (22), one can put q in any one of the equations (22)-(26) and see the contradiction with local realism in the same way.

In the straight forward way one can generalize the above Cabello like argument for two spin-s system as

$$P(s_a = s_i, s_b = s_j) = q \tag{28}$$

$$P(s_a \neq s_i, s_{b'} = s_l) = 0 \tag{29}$$

$$P(s_{a'} = s_k, s_b \neq s_j) = 0 \tag{30}$$

$$P(s_{a'} = s_k, s_{b'} = s_l) = p \tag{31}$$

where s_i, s_j, s_k and s_l can take values from $-s$ to $+s$ and $q < p$. Argument will run in the same way to show contradiction with local realism and things will remain same in which ever equations of (28) to (30) we put q on the right hand side.

States which satisfy above equations $(28) - (31)$ has to be orthogonal to the $4s$ linearly independent vectors corresponding to the projection operator appearing in equations (29) - (30) and non-orthogonal to the projection operator $P[|s_a = s_i, s_b = s_j\rangle]$ and $P[|s_{a'} = s_k, s_{b'} = s_l\rangle]$. Let M be the closed subspace generated by these $4s$ vectors and let $\overline{M}$ be the orthogonal complement to M. The dimension of $\overline{M}$ is obviously $4s^2 + 1$. The state vector in $\overline{M}$ must be orthogonal to the above $4s$ linearly independent vectors corresponding to the projection operator appearing in equations (29) and (30) and non-orthogonal to the projection operator $P[|s_a = s_i, s_b = s_j\rangle]$ and $P[|s_{a'} = s_k, s_{b'} = s_l\rangle]$. Because the dimension of $\overline{M}$ is $4s^2 + 1$ which is greater than 1, there are infinitely many vectors satisfying equations $(28) - (31)$. Then mixture of them also satisfies equations $(28) - (31)$ which shows Cabello like nonlocality. Here it is interesting to note that for $s = 1/2$, where standard Hardy's non-locality argument for given choice of observables offers a unique state as solution [6], there are more than one vector as solution. So there are mixed states even for two qubits which exhibit this kind of non-locality [8]. In straight forward way one can extend Cabello like nonlocality argument for n ($n \geq 2$) number of spin-s system.

Conclusion

Hardy's version of Bell's theorem has been considered to be interesting as it does not use inequality explicitly [11]. In higher dimension, construction of Bell's inequality remains a tough job barring some results [12]. In this context various kind of Hardy like argument can be constructed easily in higher dimensional system which may be very useful to reveal non-locality of non-trivial density matrix in higher dimension. Here we reveal the unnecessary restriction of the previous version of Hardy's non-locality in higher dimension and show that the solution set is larger than one previously found. We also discuss various generalization in the context of new forms of non-locality argument as well as various cases where more than two particles are involved.

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